

# Further development of positive semidefinite solutions of the operator equation $\sum_{j=1}^n A^{n-j} X A^{j-1} = B^*$

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**Abstract** In [7], T. Furuta discusses the existence of positive semidefinite solutions of the operator equation  $\sum_{j=1}^n A^{n-j} X A^{j-1} = B$ . In this paper, we shall apply Grand Furuta inequality to study the operator equation. A generalized special type of  $B$  is obtained due to [7].

**Keywords:** Furuta inequality; Grand Furuta inequality; operator equation; matrix equation; positive semidefinite operator; positive definite operator

**Mathematics Subject Classification:** 15A24, 47A62, 47A63.

## 1 Introduction

A capital letter  $T$  means a bounded linear operator on a Hilbert space.  $T \geq 0$  and  $T > 0$  mean a positive semidefinite operator and a positive definite operator, respectively.

In the middle of last century, E. Heinz et al. studied operator theory and obtained the following famous theorem:

**Theorem LH (Löwner-Heinz inequality, [9] [8]).** If  $A \geq B \geq 0$ , then  $A^\alpha \geq B^\alpha$  for any  $\alpha \in [0, 1]$ .

In 1987, T. Furuta proved the following operator inequality as an extension of Theorem LH:

**Theorem F (Furuta inequality, [4]).** If  $A \geq B \geq 0$ , then for each  $r \geq 0$ ,

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}} \quad (1.1)$$

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$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \quad (1.2)$$

hold for  $p \geq 0$ ,  $q \geq 1$  with  $(1+r)q \geq p+r$ .

K. Tanahashi, in [10], proved the conditions  $p$ ,  $q$  in Theorem F are best possible if  $r \geq 0$ .

In 1995, T. Furuta showed another operator inequality which interpolates Theorem F:

**Theorem GF (Grand Furuta inequality, [5]).** If  $A \geq B \geq 0$  with  $A > 0$ , then for each  $t \in [0, 1]$  and  $p \geq 1$ ,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad (1.3)$$

holds for  $s \geq 1$  and  $r \geq t$ .

Afterwards, some nice proof of Grand Furuta inequality are shown, such as [2], [6]. K. Tanahashi, in [11], proved that the outer exponent value of (1.3) is the best possible. Later on, the proof was improved, see [3], [12].

Recently, T. Furuta proved the following theorem by Furuta inequality:

**Theorem A ([7]).** If  $A$  is a positive definite operator and  $B$  is positive semidefinite operator. Let  $m$  and  $n$  be nature numbers. There exists positive semidefinite operator solution  $X$  of the following operator equation:

$$\sum_{j=1}^n A^{n-j} X A^{j-1} = A^{\frac{nr}{2(m+r)}} \left( \sum_{i=1}^m A^{\frac{n(m-i)}{m+r}} B A^{\frac{n(i-1)}{m+r}} \right) A^{\frac{nr}{2(m+r)}} \quad (1.4)$$

for  $r$  such that  $\begin{cases} r \geq 0, & \text{if } n \geq m; \\ r \geq \frac{m-n}{n-1}, & \text{if } m \geq n \geq 2. \end{cases}$

In the rest of this short paper, we shall apply Grand Furuta inequality to discuss the existence of positive semidefinite solution of operator equation  $\sum_{j=1}^n A^{n-j} B A^{j-1} = B$ , and show a generalized special type of  $B$  due to Theorem A.

## 2 Extension of Furuta's result

**Lemma 2.1** ([1], [7]). Let  $A$  be a positive definite operator and  $B$  a positive semidefinite operator. Let  $m$  be a positive integer and  $x \geq 0$ . Then  $\frac{d}{dx}[(A + xB)^m] \Big|_{x=0} = \sum_{j=1}^m A^{m-j} B A^{j-1}$ .

**Theorem 2.1.** Let  $A$  be a positive definite operator and  $B$  be a positive semidefinite operator. Let  $m, n, k$  be positive integers,  $t \in [0, 1]$ . There exists positive semidefinite operator solution  $X$  which satisfies the operator equation:

$$\sum_{j=1}^n A^{n-j} X A^{j-1} = A^{\frac{nr}{2(m-t)k+2r}} \left\{ \sum_{i=1}^k A^{\frac{n(m-t)(k-i)}{(m-t)k+r}} \left[ A^{-\frac{t}{2} \cdot \frac{n}{(m-t)k+r}} \cdot \left( \sum_{j=1}^m A^{\frac{n(m-j)}{(m-t)k+r}} B A^{\frac{n(j-1)}{(m-t)k+r}} \right) \cdot A^{-\frac{t}{2} \cdot \frac{n}{(m-t)k+r}} \right] A^{\frac{n(m-t)(i-1)}{(m-t)k+r}} \right\} A^{\frac{nr}{2(m-t)k+2r}} \quad (2.1)$$

for  $r$  such that  $\begin{cases} r \geq t, & \text{if } (1-t)n \geq (m-t)k; \\ r \geq \max\{\frac{(m-t)k-(1-t)n}{n-1}, t\}, & \text{if } (m-t)k \geq (1-t)n \text{ with } n \geq 2. \end{cases}$

**Proof.** As in the proof of [[7], Theorem 2.1], by  $A + xB \geq A > 0$  holds for any  $x \geq 0$ , then  $A^{-1} \geq (A + xB)^{-1} > 0$ . Replace  $A$  by  $A^{-1}$ ,  $B$  by  $(A + xB)^{-1}$ ,  $p$  by  $m$ ,  $s$  by  $k$  in (1.3), and take reverse, we have

$$(A^{\frac{r}{2}} (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}})^{\frac{1-t+r}{(m-t)k+r}} \geq A^{1-t+r}. \quad (2.2)$$

For any  $\alpha \in [0, 1]$ , apply Löwner-Heinz inequality to (2.2), and take  $\frac{1}{n} = \frac{1-t+r}{(m-t)k+r} \cdot \alpha$ , the following inequality is obtained:

$$(A^{\frac{r}{2}} (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}})^{\frac{1}{n}} \geq A^{\frac{(m-t)k+r}{n}}. \quad (2.3)$$

By  $\alpha \in [0, 1]$  and the condition of  $r$  in Grand Furuta inequality, we can take  $r \geq t$  if  $(1-t)n \geq (m-t)k$ , or  $r \geq \max\{\frac{(m-t)k-(1-t)n}{n-1}, t\}$  if  $(m-t)k \geq (1-t)n$  with  $n \geq 2$ .

Take  $Y(x) = (A^{\frac{r}{2}} (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}})^{\frac{1}{n}}$ . According to (2.3),  $Y(x) \geq Y(0) = A^{\frac{(m-t)k+r}{n}}$  for any  $x \geq 0$ , then  $Y'(0) \geq 0$ . Differentiate  $Y^n(x) = A^{\frac{r}{2}} (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}}$ , use Lemma 2.1, then take  $x = 0$ , the following equality holds:

$$\begin{aligned} \frac{d}{dx} [Y^n(x)] \Big|_{x=0} &= \sum_{j=1}^n Y(0)^{n-j} Y'(0) Y^{j-1} \\ &= \frac{d}{dx} [A^{\frac{r}{2}} (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}}] \Big|_{x=0} \\ &= A^{\frac{r}{2}} \left\{ \sum_{i=1}^k \left[ (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})^{k-i} \Big|_{x=0} \right] \cdot \left[ (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})' \Big|_{x=0} \right] \right. \\ &\quad \left. \cdot (A^{-\frac{t}{2}} (A + xB)^m A^{-\frac{t}{2}})^{i-1} \Big|_{x=0} \right\} A^{\frac{r}{2}} \\ &= A^{\frac{r}{2}} \left\{ \sum_{i=1}^k [A^{(m-t)(k-i)} (A^{-\frac{t}{2}} (\sum_{j=1}^m A^{m-j} B A^{j-1}) A^{-\frac{t}{2}}) A^{(m-t)(i-1)}] \right\} A^{\frac{r}{2}}. \end{aligned}$$

Replace  $Y(0)$  by  $A^{\frac{(m-t)k+r}{n}}$ ,  $Y'(0)$  by  $X$ , we have

$$\begin{aligned} & \sum_{j=1}^n A^{\frac{(m-t)k+r}{n}(n-j)} X A^{\frac{(m-t)k+r}{n}(j-1)} \\ &= A^{\frac{r}{2}} \left\{ \sum_{i=1}^k A^{(m-t)(k-i)} [A^{-\frac{t}{2}} \left( \sum_{j=1}^m A^{m-j} B A^{j-1} \right) A^{-\frac{t}{2}}] A^{(m-t)(i-1)} \right\} A^{\frac{r}{2}}. \end{aligned}$$

Replace  $A$  by  $A^{\frac{n}{(m-t)k+r}}$  in above equality, then (2.1) is obtained.  $\square$

**Remark 2.1.** If take  $t = 0$  and  $k = 1$  in Theorem 2.1, this theorem is just Theorem A, which is the main result of [7].

**Example 2.1.** We use the same example as [7]: For two  $l \times l$  matrices  $A$  and  $B$ , take  $A = \text{diag}(a_1, a_2, \dots, a_l)$ , all entries of  $B$  are 1. If  $m, n, k$  are positive integers,  $t \in [0, 1]$ , there exists positive semidefinite matrix  $X$  which satisfies:

$$\begin{aligned} & \sum_{j=1}^n A^{\frac{(m-t)k+r}{n}(n-j)} X A^{\frac{(m-t)k+r}{n}(j-1)} \\ &= A^{\frac{r}{2}} \left\{ \sum_{i=1}^k A^{(m-t)(k-i)} [A^{-\frac{t}{2}} \left( \sum_{j=1}^m A^{m-j} B A^{j-1} \right) A^{-\frac{t}{2}}] A^{(m-t)(i-1)} \right\} A^{\frac{r}{2}} \end{aligned}$$

for  $r$  such that  $\begin{cases} r \geq t, & \text{if } (1-t)n \geq (m-t)k; \\ r \geq \max\{\frac{(m-t)k-(1-t)n}{n-1}, t\}, & \text{if } (m-t)k \geq (1-t)n \text{ with } n \geq 2. \end{cases}$

It is easy to calculate the expression of  $X$ :

$$X = \left( \frac{a_p^{\frac{r-t}{2}} a_q^{\frac{r-t}{2}} \left( \sum_{i=1}^k a_p^{(m-t)(k-i)} a_q^{(m-t)(i-1)} \right) \left( \sum_{j=1}^m a_p^{m-j} a_q^{j-1} \right)}{\sum_{j=1}^n a_p^{\frac{((m-t)k+r)(n-j)}{n}} a_q^{\frac{((m-t)k+r)(j-1)}{n}}} \right)_{p,q=1,2,\dots,l}. \quad (2.4)$$

**Remark 2.2.** The condition of  $r$  in Theorem 2.1 is necessary. If the condition cannot be fulfilled, the solution of the equation may be not positive semidefinite.

For example, take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and  $m = 2, n = 2, k = 2, t = \frac{1}{2}$  in Example 2.1. If we put  $r = \frac{1}{2}$ , then  $r \not\geq \max\{\frac{(m-t)k-(1-t)n}{n-1}, t\}$ . By (2.4), the solution of the following matrix equation

$$\begin{aligned} A^{\frac{7}{4}} X + X A^{\frac{7}{4}} &= A^{\frac{1}{4}} \left( A^{\frac{3}{2}} (A^{\frac{3}{4}} B A^{-\frac{1}{4}} + A^{-\frac{1}{4}} B A^{\frac{3}{4}}) + (A^{\frac{3}{4}} B A^{-\frac{1}{4}} + A^{-\frac{1}{4}} B A^{\frac{3}{4}}) A^{\frac{3}{2}} \right) A^{\frac{1}{4}} \\ &= A^{\frac{5}{2}} B + A^{\frac{3}{2}} B A + A B A^{\frac{3}{2}} + B A^{\frac{5}{2}} \\ &= \begin{pmatrix} 4 & 3 + 6 \times 2^{\frac{1}{2}} \\ 3 + 6 \times 2^{\frac{1}{2}} & 16 \times 2^{\frac{1}{2}} \end{pmatrix} \end{aligned}$$

is

$$X = \begin{pmatrix} 2 & \frac{3+6 \times 2^{\frac{1}{2}}}{1+2 \times 2^{\frac{3}{4}}} \\ \frac{3+6 \times 2^{\frac{1}{2}}}{1+2 \times 2^{\frac{3}{4}}} & 2 \times 2^{\frac{3}{4}} \end{pmatrix}.$$

However,  $X$  is not a positive semidefinite matrix because its eigenvalues are  $\{5.4007 \dots, -0.0372 \dots\}$ .

**Remark 2.3.** In [1], the authors showed that if  $A$  and  $Y$  are positive semidefinite matrices in matrix equation  $A^{n-1}X + A^{n-2}XA + \dots + AXA^{n-2} + XA^{n-1} = Y$ , then so is  $X$ . By Theorem 2.1, in some special cases, if  $Y$  can be expressed as the right hand of (2.1), though it is not positive semidefinite, then there still exists positive semidefinite solution satisfies  $A^{n-1}X + A^{n-2}XA + \dots + AXA^{n-2} + XA^{n-1} = Y$ .

For example, take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \times 2^{\frac{1}{3}} \end{pmatrix}, Y = \begin{pmatrix} 4 & 3 \times 2^{\frac{1}{4}} + 6 \times 2^{\frac{3}{4}} \\ 3 \times 2^{\frac{1}{4}} + 6 \times 2^{\frac{3}{4}} & 32 \end{pmatrix}.$$

Although  $Y$  is not a positive semidefinite matrix (because its eigenvalues are  $\{37.5589 \dots, -1.5589 \dots\}$ ), by simple calculation, the solution of the following matrix equation

$$A^2X + AXA + XA^2 = Y$$

is

$$X = \begin{pmatrix} \frac{4}{3} & \frac{3 \times 2^{\frac{1}{4}} + 6 \times 2^{\frac{3}{4}}}{1+2 \times 2^{\frac{1}{3}} + 4 \times 2^{\frac{2}{3}}} \\ \frac{3 \times 2^{\frac{1}{4}} + 6 \times 2^{\frac{3}{4}}}{1+2 \times 2^{\frac{1}{3}} + 4 \times 2^{\frac{2}{3}}} & \frac{4 \times 2^{\frac{2}{3}}}{3} \end{pmatrix},$$

which is still a definite matrix whose eigenvalues are  $\{2.9013 \dots, 0.1119 \dots\}$ . The critical reason is that  $Y$  can be expressed as follows,

$$Y = A^{\frac{3}{8}} \left\{ \sum_{i=1}^2 A^{\frac{9}{8}(2-i)} \left[ A^{-\frac{3}{16}} \left( \sum_{j=1}^2 A^{\frac{3}{4}(2-j)} B A^{\frac{3}{4}(j-1)} \right) A^{-\frac{3}{16}} \right] A^{\frac{9}{8}(i-1)} \right\} A^{\frac{3}{8}},$$

which is the right hand of (2.1) under the condition of  $m = 2, n = 3, k = 2, t = \frac{1}{2}, r = 1$ .

**Remark 2.4.** The following question remains open: How to investigate the properties of the solution of operator equation  $X^{n-1}A + X^{n-2}AX + \dots + XAX^{n-2} + AX^{n-1} = B$ ?

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